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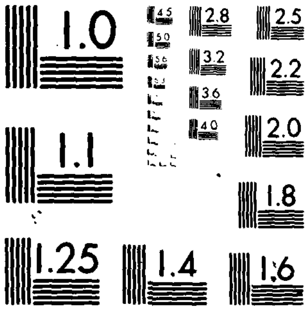

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AN ELEMENTARY EXPOSITION OF THE HARSANYI—SELTEN TRACING PROCEDURE

Final Report

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J. Harsanyi and R. Selten have recently developed a new theory that attempts to determine a unique solution to any game. In Chapter 1 the historical and intellectual background of the new Harsanyi-Selten "tracing procedure" is given. In Chapter 2, the author presents an elementary exposition of the tracing procedure itself. Finally in CHapter 3, the author presents a worked sample application of the tracing procedure in the form of an analysis of a one parameter family of two-person bargaining games. Throughout, there is an emphasis on the intuitive aspects of the new theory.

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1. INTRODUCTION AND OVERVIEW

During the past eight years, John Harsanyi and Reinhard Selten have collaborated on the development of a new approach to game theory. This approach centers upon an operation they call the "tracing procedure". Their new theory is at once very complex, very general and very ambitious: it attempts nothing less than to prescribe a unique solution to any and every competitive situation that can properly be called a "game".

In this research paper, we attempt to describe the tracing procedure and to motivate it by applying it to a simple worked example. However, before proceeding to the intricacies of the Harsanyi-Selten theory, we shall attempt here in Chapter 1 to put the new research effort in proper perspective. We ask and partially answer such queries as: Why have Harsanyi-Selten been prompted to develop the theory? What predecessor theories were there, and what was wrong with them? Is the new theory likely to be useful in practical applications? In Section 1.1 we discuss two foundational problems in game theory that prompted the new theory. In 1.2 we first (Section 1.2.A) discuss an earlier theory that Harsanyi developed and then abandoned, the so-called bargaining equilibrium model for non-cooperative games. Then in 1.2.B we contrast the new tracing theory with the older theory. In Section 1.3 we argue that the new theory is not likely to be useful in practical applications, for reasons having to do not only with the theory but with the purpose of game theory itself.

Subsequently, in Chapter 2, we offer a somewhat technical description of the tracing procedure itself. This second chapter also contains a philosophically oriented appraisal of the procedure that emphasizes its decision theoretical and game theoretical interpretations. Then in Chapter 3 we furnish a worked application analyzed in recent unpublished work by Harsanyi (and also by Guth and Selten). A two-person bargaining game with incomplete information is constructed. It is shown that the tracing procedure yields a solution that is almost identical to the classical two-person Nash bargaining solution of a game of complete information that is almost identical to the example being analyzed. This result is of interest not only because of the historical interest and importance of the Nash bargaining solution, but also because of the well-known intuitive appeal of the solution.

1.1. Two Foundational Problems in the Theory of Games

While there are several fundamental "difficulties" with game theory as a mathematical discipline, two problems stand out as fundamental. First, there is the somewhat arbitrary and obfuscatory division of competitive situations into two different classes: "cooperative" and "non-cooperative". Classical game theory follows Nash in defining cooperative games as those permitting both free communication between the players and also enforceable agreements between them. All other games are non-cooperative, although classical game theory usually assumes that a non-cooperative game is characterized by an absence both of binding agreements and communication. The second fundamental problem is that of defining a unique solution to games of various types. Concerning solution theory, the ideal has been to do what von Neumann originally did in his original 1928 paper on zero-sum games: namely to characterize a unique solution for all games of the class under consideration. Regrettably, with the exception of the class of zero-sum games and certain two-person cooperative games (e.g. the Nash bargaining games), there are very few classes of games for which determinate solution theories have been formulated. And to the extent that determinate theories have been set forth (e.g. the Shapley value for n-person transferable utility games) there is an even deeper problem: Why should different classes of games require fundamentally different solution theories, each with a different "logic"? Wouldn't it be preferable to have a single overarching solution theory that would be based on a single set of principles applying to all games whatsoever?

1.2. The New Harsanyi-Selten Theory

The new Harsanyi-Selten theory attempts to solve both of the foundational problems cited above in one fell swoop. It takes the concept of a non-cooperative game as basic, and essentially requires that any given cooperative game to be analyzed be reformulated as a non-cooperative game. The theory then restricts its attention to the set of equilibrium points of the non-cooperative game, and attempts to select one of these equilibrium points as the solution to the game. Exactly how this selection process operates is what we shall be discussing in subsequent chapters. In the present chapter, we shall merely sketch the spirit of the process. Before doing this, however, it will be helpful to review the only other approach to defining a unique solution to all games -- John Harsanyi's "Bargaining Equilibrium" theory set forth in 1977 in Harsanyi (5).*

1.2.A. The Bargaining Equilibrium Solution Theory

Harsanyi's earlier work will be of interest to us not only because his theory was the first theory to attempt to define a unique solution to all games based on a single set of principles, but also because it made use of a concept ("risk dominance") that plays a modified role in the newer Harsanyi-Selten theory.

In 1950, John F. Nash, Jr. (8) had axiomatized a unique solution to the two-person bargaining game with complete information. While Nash's model was mathematically very elegant and abstract, it failed to provide any psychologically convincing reasons as to why rational bargaining in the real world would result in the "solution" prescribed by Nash's theory. Harsanyi (2) remedied this deficiency in an

* While Harsanyi did not publish his bargaining equilibrium approach in book form until 1977, his work was largely completed by 1970. And it was around 1970 that Harsanyi joined Selten in the newer Harsanyi-Selten theory that is being discussed in the present paper.

important 1956 paper. Building upon and extending an earlier model of F. Zeuthen (11), he views bargaining as a dynamic process in which the antagonists make "demands" and "concessions". Indeed, suppose that we are at stage k in a two-person bargaining game. Let U_1 and U_2 be the bargainers' respective utility functions. Denote their last offers at stage k as ${A_1}^k = A_1$ and ${A_2}^k = A_2$. The critical question is: Which player will have to make the next concession at stage k+1? The Harsanyi-Zeuthen argument is that the next concession must come from the player who is less willing to face the risk of a conflict. (In the model, the "conflict situation", denoted by C, represents what happens if negotiations collapse and both players go away with nothing.) But how can we measure a given player's willingness to risk a conflict rather than to accept an opponent's terms? In explaining how the Harsanyi-Zeuthen theory answers this critical question, I shall follow Harsanyi (5 , pp. 150-155).

Because we want to measure each player's willingness to stick to his own terms rather than to accept his antagonist's terms, we can view player i's decision as a choice between two alternatives, a' and b'. Here a' denotes full insistence on i's own last offer A_i, and b' denotes full acceptance of his opponent's last offer A_j. Assuming i is "rational" (i.e. that he is a Bayesian expected utility maximizer) then he must start by assigning subjective probabilities to the two possible choices that his opponent can make in reply. Let p_{ji} be the subjective probability that i assigns to the hypothesis that j will chose alternative a'; and let $q_{ji} = 1 - p_{ji}$ denote the subjective probability that i assigns to the hypothesis that j will choose b'.

If i himself chooses b', then according to the Harsanyi-Zeuthen model, he will obtain payoff $U_i(A_j)$ with certainty, regardless of what j does. On the other hand, if i chooses alternative a', then he may obtain the higher payoff $U_i(A_i) > U_i(A_j)$; but he may also obtain the conflict payoff $U_i(C) < U_i(A_j)$. The former outcome will occur with probability q_{ji} whereas the latter will occur with probability p_{ji}.

Thus if player i wants to maximize his expected utility, then he will stand on his own last offer A_i, *that is select* a', only if

$$(1 - p_{ji}) \cdot U_i(A_i) + p_{ji} \cdot U_i(C) \geqslant U_i(A_j) \qquad (1.1)$$

Equivalently, i will select a' only if

$$p_{ji} \leqslant r_i = \frac{U_i(A_i) - U_i(A_j)}{U_i(A_i) - U_i(C)} \qquad (1.2)$$

The quantity r_i $(i = 1,2)$ defined in (1.2) is called player i's risk limit since it represents the highest risk (the highest subjective probability of a conflict) that player i would be willing to face in order to achieve an agreement on his own terms A_i (rather than on his opponent's terms A_j). This is so because, if player i sticks to his own last offer A_i, then he must expect a conflict to occur with probability p_{ji} (this is the probability that his opponent will stick to his own last offer A_j, in which event neither makes a concession and a conflict results). But, from (1.2) we know that the highest value of the probability p_{ji} that i can face, without switching over to acceptance of the opponent's last offer, is $p_{ji} = r_i$.

We have set forth the basic logic of the Harsanyi-Zeuthen scheme. Several important points should be added to the foregoing sketch of their model. First, Harsanyi shows that each player i will always know the subjective probabilities p_{ji} and q_{ji} he needs in performing his "utility maximization" according to the above logic. We do not have space to show why this is so, but it is important since it offers one of the few cases where "subjective" probabilities are in fact quasi-"objective". Second, the Harsanyi-Zeuthen model implies that the bargaining process will stop -- and agreement will be reached -- at that point where the arithmetic product of the players' utilities is reached. But this is the well-known prescription of Nash's abstract model. Hence the two models imply the same bargaining "solution", namely the set of terms whereby the product-of-the-utilities is maximized. We shall not work out the details of this equivalence here, but refer the reader to the book by Harsanyi.

Third, note that we can apply our analysis to j as well as to i, and we can compute the quantity r_j (as well as r_i) when evaluating any set of bargaining terms. Thus, we can define a risk dominance concept along the following lines. If $r_i < r_j$, then we know that player i is less willing than player j to risk a conflict, or equivalently, that he has less incentive to do so. But both players will know this. Therefore player i will be under strong psychological pressure to make the next concession, whereas player j will be unwilling to do so. This is the essence of the concept of risk dominance that in one form or another plays an important role both in Harsanyi's first general theory (described just below) of 1977, and in the later Harsanyi-Selten theory. We shall now briefly discuss the essence of Harsanyi's "bargaining equilibrium approach" to defining a unique solution to all games.

In discussing the problem of multiple equilibrium points and the need for a determinate solution theory in game theory, Luce and Raiffa (7, pp. 109-110) present the following non-cooperative game:

	B_1	B_2
A_1	(4, -30)	(10, 6)
A_2	(8,8)	(5,4)

Here player 1 has a choice between strategies A_1 and A_2. Player 2 has a choice between B_1 and B_2. The two pairs of strategies (A_2,B_1) and (A_1,B_2) are both equilibrium points of the game: that is, (i) A_2 is a best reply strategy to B_1 just as B_1 is a best reply to A_2, and (ii) A_1 is optimal against B_2 just as B_2 is optimal against A_1. We see here an instance of perhaps the most fundamental problem in game theory, that of the multiplicity of equilibrium points in non-cooperative games. In the present case, is there some way of saying that (A_1,B_2) is "better than" or "worse than" (A_2,B_1)?

Luce and Raiffa argue as follows. If player 2 has any reason to fear that 1 will take A_1, then he dare not take B_1 for fear of getting -30. But player 1, knowing this, has every reason to take A_1 which gives him his maximum payoff. But now the argument is cyclical. For 2, having some rationalization for 1's adoption of A_1 has all the more reason to avoid B_1. Thus the equilibrium point (A_1,B_2) psychologically dominates (A_2,B_1).

Luce and Raiffa's suggestion here is merely a suggestion, and a rather imprecise one at that. They have not really made clear either when psychological dominance will occur (in some games there may be no psychologically dominating equilibrium points) or what the exact nature of this dominance really is.

What Harsanyi did in his first general solution theory (his "bargaining equilibrium approach") is to show that (i) the Harsanyi-Zeuthen concept of risk dominance originating in two-person cooperative bargaining games can be used to give a precise quantitative meaning to Luce and Raiffa's "psychological dominance" concept; and (ii) the risk dominance concept can provide the basis for a determinate solution theory that identifies a unique equilibrium point as the solution to any given non-cooperative game. To see how Harsanyi's first theory works, let us consider the following simple example set forth by Harsanyi (5, pp. 275-276). Consider the following non-cooperative game:

	B_1	B_2
A_1	(2,1)	(-10,-1)
A_2	(0,0)	(1, 2)

In this game there are only two equilibrium points, namely $s^1 = (A_1,B_1)$ and $s^2 = (A_2,B_2)$*. Obviously player 1 will prefer s^1 whereas player 2 will prefer s^2. If both players use their strategies associated with their own preferred equilibrium points, then player 1 would suffer a

* There is a third equilibrium point, namely $(1/2\ A_1 + 1/2A_2,\ 11/13B_1 + 2/13B_2)$. But for rather technical reasons, Harsanyi rules this equilibrium point out of consideration in his theory. It is "ineligible".

much greater loss (in relation to the payoff difference for him between the two equilibrium points) than player 2: for $U_1(A_1,B_2) = -10$, while $U_2(A_1,B_2) = -1$. Hence both players will know that player 1 will be more fearful to use strategy A_1 in defiance of player 2's using B_2 than player 2 will be fearful to use B_2 in defiance of player 1's using A_1. Thus we see that, in Luce and Raiffa's terminology, both players will settle down at equilibrium point $s^2 = (A_2,B_2)$ preferred by player 2 since this point psychologically dominates point s^1.

Harsanyi restates this point in terms of risk dominance as follows. He views the choice between the two eligible equilibrium points as a kind of tacit bargaining game between the two players. Assuming each player attempts to maximize his expected utility, then, for reasons we have seen, player 1 will stick to strategy A_1 as long as the subjective probability that he attaches to the hypothesis that player 2 will use strategy B_2 is no greater than

$$r_1 = \frac{U_1(A_1,B_1) - U_1(A_2,B_2)}{U_1(A_1,B_1) - U_1(A_1,B_2)} = \frac{1}{12} = .08 \qquad (1.3)$$

Likewise player 2 will stick to strategy B_2 as long as the subjective probability that he attaches to the hypothesis that player 1 will stick to A_1 is no greater than his risk limit r_2. r_2 is easily calculated to be .33. By the Harsanyi-Zeuthen concession principle, since $r_1 = .08$ is smaller than $r_2 = .33$, player 1 must yield and accept as the solution to the game the equilibrium point $s^2 = (A_2,B_2)$ preferred by his opponent. In brief, we say that s^2 risk dominates s^1.

Harsanyi's general theory for defining a solution to any n-person non-cooperative game is an elaborate generalization of the ideas we have seen worked out in the above example. As we have argued in a recent review of Harsanyi's 1977 book (5 , pp. 358-361), this bargaining equilibrium solution theory is only a partial success. There are two basic reasons for this judgment, and these views are shared by Harsanyi. First, the theoiy is too complex. If the theory boiled down simply to establishing a linear ordering of the different eligible equilibrium points of a given game by using the generalized risk dominance relations Harsanyi develops, then all would be well. But this is not the case. In order to get a determinate theory, Harsanyi finds himself to classify the various equilibrium points of a game as "weak", "strong", "centroid", "weakly/strongly efficient", "inefficient", "admissible", and so on. It is all too complex. Second, there are cases where the players in Harsanyi's theory will end up using strategies that do not constitute an equilibrium point. They are obliged rather to use maximin strategies. This is unsatisfactory since, as Harsanyi usually insists, any "rational" solution to any game must at a minimum be an equilibrium point of the game.

1.2.B. The New Harsanyi-Selten Theory

Prompted in part by the criticisms we have just advanced of Harsanyi's bargaining equilibrium solution theory, Harsanyi and Selten proceeded to develop a different determinate solution theory. Since an operation called the "tracing procedure" lies at the heart of this theory, the entire theory is often spoken of as the "tracing procedure theory".

Since we are about to examine the tracing procedure in considerable detail, the only thing we shall attempt here is to suggest how this new theory contrasts with the older bargaining equilibrium model of Harsanyi. As we see it, there are two distinctive differences between the two approaches. First, the bargaining equilibrium approach has a distinctive cooperative game spirit about it. To begin with, the games analyzed by Nash and by Harsanyi-Zeuthen in their papers on the bargaining problem were cooperative games. Moreover, the Harsanyi-Zeuthen model of bargaining as a process of demands and concessions resulting in a binding agreement is far more suitable to cooperative games (in which binding agreements and communication are possible) than to non-cooperative games. In contrast to this, the tracing procedure theory is distinctively non-cooperative in spirit. According to this theory, any game whatsoever that is to be "solved" is reduced to a non-cooperative game. Thereafter, the solution theory operates solely on the equilibrium points of the game. And in selecting one equilibrium point as the "solution" to the game, the tracing procedure does not rely upon a principle of "tacit bargaining" or anything else that smacks of classical cooperative games.

The second fundamental difference between the old and the new solution theory lies in the psychological model that is developed to yield a determinate solution. The bargaining equilibrium model makes use of the concept of psychological dominance, or more specifically risk dominance. As we have seen, symmetrically rational players will, through a process of tacit bargaining, end up playing the strategies of that equilibrium point that risk dominates all other equilibrium points. In the tracing procedure theory, the model that brings symmetrically rational players into agreement as to which equilibrium point strategies to play is both cognitive and psychological. At this point in this essay, it is difficult to explain exactly what we mean by this assertion. Briefly, during the operation of the tracing procedure, the players start off by assigning prior probabilities to each other's behavior. These prior probabilities are based on what Harsanyi and Selten call "naive Bayesian rationality". Specifically, the theory makes the prior probability that player i will use strategy s_i as opposed to s_i' proportional to his net payoffs associated with the two equilibrium points s and s'. This form of rationality is "naive" because it turns out that the strategies recommended to each player via this approach do not constitute an equilibrium point of the underlying game being analyzed. But a "solution" must (according to game theory) be an equilibrium point. Thus it is necessary for the players to "modify" the naive behavior in such a way that they end up agreeing on a genuine equilibrium point as the solution. But how do they modify their initial, naive behavior? They do so by viewing the information contained in their initial strategy inclinations as "prior" information that is "updated" according to the rather complex mechanics of the tracing procedure itself (cf. Chapter 2). When this updating is completed, a complete convergence of strategic intentions and expectations has been achieved, and all the players tacitly agree which equilibrium point is the solution. During this process of updating, we shall observe that certain players "switch" from their initial (naive) strategies to the strategies corresponding to other equilibrium points. The reason why switching takes place, and the order in which the various players do (or do not) switch, is explained in terms of a new concept of risk dominance that Harsanyi-Selten introduce. We shall observe this in detail in Chapter 3.

Thus, the tracing procedure solution theory contains a rich cognitive model concerning the manner in which the players in a game form preliminary expectations about each other's behavior, and subsequently modify these expectations via informational updating. But the updating process relies in part upon a rather psychological model of risk dominance instantiated by the strategy-switching behavior of the players as they grope their way in the dark to a "solution" that is tacitly agreed upon by all. Thus it is that the new theory differs from the bargaining equilibrium theory in containing a strong cognitive component that is lacking in the former. All this will become more clear in the sequel.

Hopefully, the foregoing remarks serve to place in a rather broad perspective the relationship bet' .en the new Harsanyi-Selten theory and the only other previous attempt to define a determinate solution theory applicable to all games.

1.3 The Applicability of the New Theory to Real-World Problems

One question that immediately arises when discussing such a rich, general and complex theory as the Harsanyi-Selten theory is whether it is likely to be of "practical" interest. Our present and rather tentative belief is that the Harsanyi-Selten theory is not likely to prove to be of practical importance. The reason for this lies partly in the nature of the new theory itself, and partly in the nature and purpose of game theory as a whole. Let us begin with the second of these two reasons.

It is frequently asserted that game theory assumes that people are perfectly rational. But as Oskar Morgenstern pointed out to the present writer on several occasions, this is a misconception.

The fundamental purpose of game theory -- as well as its greatest contribution to date -- lies in its success in making clear what it means for people to behave "rationally" in game situations. The theory is prescriptive and philosophical in nature. But herein lies one of the limitations of game theory as regards practical applications. People are not in fact perfectly rational. Hence game theory is not particularly useful for predicting what people will do in a given competitive situation. Let us explore this difficulty at a somewhat deeper level by contrasting the potential usefulness of decision theory on the one hand, and game theory on the other hand.

Suppose there are n players who are involved in a competitive situation, loosely, a "game". Let us take the vantage point of player i in deciding whether or not to use game theory, decision theory, or neither in determining a best course of action. (Player i might denote the United States in a world of n military states that interact). If i uses decision theory, he will be able to determine a strategy that is rational for him regardless of whether he believes any or all of his antagonists to be rational. More pertinently, if he has reason to suppose that any or (more typically) all of his antagonists are irrational, he can bring this information to bear on his own strategy selection problem in the following manner. He can use the information when assigning his prior probabilities to the actions of his antagonists. Then, given these priors, and given any other (structural) information that he possesses, he can use decision theory to determine a course of action that is optimal for himself. Thus, decision theory is highly flexible: it can be used by party i in a wide variety of cases, most importantly in those cases where he has reason to believe that his antagonists are not perfectly rational in a game theoretical sense. Note that in stating this we are not saying how player i can defensibly choose his prior probabilities concerning his antagonists' action. This is a notoriously difficult problem, and is one of the problems that gave rise to the theory of games in the first place.

If player i wishes to use game theory rather than decision theory he will immediately run into two problems. First, game theory instructs him not to use game theory unless he is sure that all (n-1) antagonists are perfectly rational in a game theoretic context. The reason for this is that game theory is a prescriptive theory of symmetrically rational behavior that assumes that all players have access to and follow the rules of "game theory". And here we run into the second problem. There are many different versions of game theory ; in particular, there are a host of rival definitions of what constitutes a "solution". In short, even if all n players wish to act perfectly rationally and expect each other to do so, they would have no way of telling which "brand" of game theory each other subscribes to. The upshot of all this is that game theory is as inflexible and useless for practical applications as Bayesian decision theory is flexible and useful.

The second reason why the new Harsanyi-Selten theory is not likely to prove useful in practice is related to the first reason. Even if all n players in our fictitious game wished to act perfectly rationally, it is most unlikely that they would identify perfect symmetric rationality with Harsanyi-Selten behavior. Moreover, even if they did all agree with the Harsanyi-Selten characterization of rationality in game situations, we feel it most unlikely that they could operationalize the theory in the manner required to actually compute a solution. Both time constraints as well as the probability of interpretative error underly this **assertion**. We suspect that the reader will come to agree with our position here after reading Chapters 2 and 3 in the sequel where the intricacies of the new theory are explored in some detail.

In advancing the foregoing criticism of the Harsanyi-Selten theory, we do not in any way intend to denigrate its ultimate significance. Indeed, this theory lies in the best tradition of game theory. For it is a notable philosophical contribution attempting to clarify what "perfect rationality" means in a very general class of game situations. Moreover, Harsanyi and Selten have tackled head-on the most difficult problem of all, namely characterizing a unique solution to <u>all</u> games.

1.4. A Note on the Content of Chapters 2 and 3

In the following chapters, we first discuss the theory of the tracing procedure. Thereafter, we draw on an example due to Harsanyi and show how the tracing procedure works to define a solution to a certain family of two-person bargaining games of incomplete information. Our intent has been to produce a paper that is intelligible by as large a cross section of readers as possible. Harsanyi has described his new theory in several articles, e.g. (3 and 4). However, anyone familiar with these expositions is aware of two problems: first, the new theory appears as extremely complicated. Second, it is difficult to penetrate the formal theory because no practical applications are given. The present treatment differs from any other we know of because it attempts to overcome both these problems in an effort to broaden the audience interested in and familiar with the theory.

Specifically, our presentation of the formal theory in Chapter 2 (and partly in Chapter 3) takes certain liberties aimed at streamlining the theory. We present only those portions of the formal theory that are needed both to communicate the basic ideas underlying the tracing procedure and to permit comprehension of the worked example of Chapter 3. This approach reduces the complexity of the presentation, permits it to be briefer and more accessible, and ensures that all the formal ideas presented are relevant to the practical example. The price paid for all this is an omission of a discussion of (i) the uniform perturbation of games in agent normal form (even though we do discuss the agent normal form), (ii) cells and formations -- concepts that while interesting and important are quite complex and play no role in the example of Chapter 3, and (iii) the discretization of the family of games G(w) analyzed in Chapter 3.

With these caveats in mind, let us now turn to the new Harsanyi-Selten theory and learn what the tracing procedure is, and how it works.

2. THE TRACING PROCEDURE

In Chapter 1 we provided both a brief description of and the motivation for the new HS "tracing procedure". In this Chapter we shall give a somewhat technical description of the tracing procedure itself. In terms of demands upon the reader, this Chapter falls midway between Chapter 1 (which is accessible to anyone with an elementary interest in game theory) and Chapter 3 (which is necessarily technical at points).

In 2.1 below, we give a somewhat precise overview of the tracing procedure, emphasizing the question of the prior probability distributions that are so important in the theory. In 2.2 we turn to the tracing procedure itself (viewed as a process that operates on a given n-tuple of prior distributions) and discuss one particular variant of it in detail: the linear tracing procedure. Finally, in 2.3 we give a rather philosophical appraisal of this procedure, emphasizing its decision theoretical and game theoretical interpretations and inspirations.

2.1 The Tracing Procedure and the Priors

As mentioned in Chapter 1, the tracing procedure is essentially a mathematical model of a process of convergent expectations entertained by the n players in any game. More specifically, the tracing procedure (henceforth, TP) models a solution process by which rational people will come to adopt, and come to expect each other to adopt, one specific equilibrium point $q^* = (q_1^*,\ldots,q_n^*)$ as the solution for a given noncooperative game G.

At the beginning of this process, the players are ignorant as to which strategies will be used by their mutual antagonists. Thus, according to generalized Bayesian logic, each player j will express his expectations about the strategy choice of any other player $i \neq j$ in the form of a subjective probability distribution $p_i = (p_{i1},\ldots,p_{iK_i})$ over i's pure strategies, where p_{ik} $(k = 1,\ldots,K_i)$ is the probability that player j assigns to the hypothesis that player i will actually use his kth pure strategy ϕ_i^k. These subjective distributions p_i are to be called the prior probability distributions or simply the priors of the TP.

One very important aspect of the HS theory is that it prescribes the prior p_i of each and every player in any game G. Additionally, the HS theory goes further than this, and it assumes that all (n-1) antagonists of player i will in fact use this prior p_i prescribed by the theory. The question of "where these priors p_i come from" will best be answered in Chapter 3 where we actually compute a family of priors in the context of a concrete family of games. Briefly, in most cases, the HS theory priors have the property that a given prior p_i will assign higher probability to those pure strategies that would be best replies to the other players' expected strategies in wider ranges of possible strategic situations.

Formally speaking, every prior p_i is a probability distribution over player i's pure strategy set. Thus it has the nature of a "mixed strategy" in game theoretic terms. However its game theoretic interpretation is very different from that of a standard mixed strategy for two specific reasons: (i) the probabilities q_{ik} associated with a true mixed strategy in game theory are objective probabilities chosen by player i himself on the basis of a solution theory, whereas the probabilities p_{ik} associated with an HS prior are subjective probabilities expressing the other players' expectations about player i's likely behavior. (ii) In a true mixed strategy, the probabilities denote the intentions of the players to deliberately randomize their pure strategies. Obviously, this is not the case in the HS theory as described above.

$p = (p_1,\dots,p_n)$ will henceforth denote the n-tuple of all priors in the game, whereas the (n-1)-tuple obtained from p via deletion of the ith component p_i will be expressed as $p_{-i} = (p_1,\dots,p_{i-1},p_{i+1},\dots,p_n)$. p will be called a prior vector and p_{-i} will be called an i-incomplete prior vector.

At this point we turn to a problem HS call the "prediction problem". Let us take as given the n players' mutual expectations about one another's behavior as given by the prior vector p. How can we now predict the actual strategy combination $q = (q_1,\dots,q_n)$ these players will use in practice? The entire purpose of the TP is to answer this problem.

At this point we come to a discussion of the dynamics of the TP itself. We have already sketched how it is possible for the players to formulate their mutual expectations about each others' likely behavior in the form of the vector p of priors. We now ask whether and when it would in fact be rational for the players, having formulated p, to rely on p for all the information needed in determining what it is rational to actually do during the game.

As indicated above, the expectations of player i concerning the other players' strategies are given by the i-incomplete prior p_{-i}. Thus it might seem that each player i will play a strategy q_i^o that is his best reply to this prior p_{-i}. Formally, he will determine a strategy $q_i = q_i^o$ that solves

$$\text{MAX} \quad H_i = H(q_i,p_{-i}) \qquad (2.1)$$

where $H(\cdot)$ is i's payoff (utility) function. Generalizing this train of thought, it might seem that the solution to the game would be the vector $q^o = (q_1^o,\dots,q_n^o)$ consisting of the set of these best replies for all n players. HS call this simplistic, first-order solution the naive Bayesian approach.

Why is this straightforward solution theory "naive"? The answer is simple: q^o will not in general be an equilibrium point of the game G. Player i's strategy q_i^o will not be a best reply to player j's strategy q_j^o, etc. As is well-known, in order for a strategy n-tuple to qualify as a bona fide solution in game theory, it must constitute an equilibrium point of a game.

It should prove fruitful to examine in more detail the ostensible conflict between "naive Bayesian rationality" on the one hand and game theoretic (equilibrium point) rationality on the other hand. In analyzing this conflict, we shall see not only the key idea underlying the tracing procedure, but also exactly why it is that HS speak of their new theory as Bayesian in spirit.

One of the central tenets in Bayesian statistics and decision theory is that people start off at some point in time t with information called their prior (or "first-order") information. This initial information can be denoted I^P_t. Then as we move to time periods $t+1$, $t+2...,t+k$, they acquire new information and hence "update" their prior information I^P_t with the new information ("second-order") they obtain at each subsequent point in time. They end up with I^P_{t+k}. In decision theory proper, the proper way to "update" one's first order information with second-order information is via Bayes rule of probability theory.

An analogue of this information updating procedure lies at the heart of HS's TP. It is for this reason that their approach is called Bayesian (also see 2.3 below). In the TP, the vector p of priors is regarded to be the players' first-order information. Any subsequent information they might receive is called second-order information. But why would there be any need for second-order information? After all, the given game G, since it is assumed to be well-defined, should contain all the information that is necessary and relevant, or should it? Let us press this question further.

HS distinguish between the specific information that is contained in the vector p -- first order information, and any additional information that the players may come to have about the likely reactions of one another to the strategy implications implicit in their first order information p. This latter type of information is the second-order information mentioned above.

To see all this more clearly, let us look at two distinct cases. First, suppose that the strategy n-tuple q^o obtained when each player solves (2.1) above does, upon reflection, turn out to be "rational" for all n players. Then this conclusion about strategic rationality is regarded as second-order information in the HS theory. This would be the case were q^o in fact an equilibrium point of the given game G.

Next suppose that q^o is not in fact an equilibrium point, and this will generally be the case. If q^o is not an equilibrium point, then by definition there must be at least one player i whose naive Bayesian strategy q_o^i is not a best reply to the strategy (n-1) tuple q_{-i}^o of the other (n-1) players. When the players realize this, they will realize that the naive Bayesian strategy q^o is "recommending" irrational behavior on the part of at least one player. More specifically, second-order information about strategic rationality has come into conflict with the first-order information summarized by p.

How is this conflict resolved? HS make the generalized Bayesian assumption that in the event of such conflicts (e.g. when q^o is not an equilibrium point) the players take their naive Bayesian information (and its implied strategic behavior) p (q^o) as a starting point. They then "update this information" with second-order, third-order,...,k^{th}-order information as this information is obtained. But exactly how does this updating take place? Is there a game theoretic analogue to Bayes' theorem that is called upon at analogous junctions in their theory? Yes: HS implicitly assert that the tracing procedure (about to be formally defined) is the appropriate analogue to Bayes rule. It serves to update p by feeding in subsequent information in a strategy updating process until such time as there remains no discrepancy at all between first and second order information.

In the following Section 2.2, the TP is formally introduced and characterized. However, we wish to emphasize that the reader who really wishes to understand how the TP works will wish to read Chapter 3 as well as Section 2.2 below.

2.2 The Linear Tracing Procedure

In Section 2.1 we have seen how the HS theory leads to the formation of prior strategic information and naive Bayesian behavior. At the beginning of their decision-making process, the players' *expectations* (first-order information) about each other's behavior are given by the prior p while their tentative *strategy plans* (first-order behavior) are given by the strategy n-tuple q^o. The TP itself is best viewed as an *operator* that takes p as given and gradually updates it by feeding back information about strategic rationality. More specifically, it is a *Bayesian operator* precisely because of its role that is analogous to the role of Bayes' operator. As the TP operates, i.e. as updating takes place, both p and q^o are subjected to systematic and continuous transformations until they both finally converge on a specific equilibrium point q^* of the given game G. Thus, at the end of the process, a solution q^* is obtained according to which there is complete agreement between the players' *actual strategy plans* and their *expectations*.

We shall now introduce the most accessible and transparent version of the TP. It is called the linear tracing procedure. It is always used in the HS theory except when mathematical complications require a more general procedure (the *logarithmic* TP) to be substituted.

The linear tracing procedure (henceforth simply TP) is based upon a one-parameter family of games $\{G^t\}$ with $0 \leq t \leq 1$. This one-parameter family of games is derived from the original given game G as follows. In any game G^t, every player i (i = 1,...,n) will have the same strategy set $Q_i = \{q_i\}$ as in the underlying game G. But his payoff function H_i^t in G^t will be different, defined now as

$$H_i^t(q_i,q_{-i}) = t\, H_i\,(q_i,q_{-i}) + (1-t)\, H_i\,(q_i,p_{-i}) \qquad (2.2)$$

where H_i is i's payoff function in the game G. When t = 1 we have

$$H_i^1\,(q_i,q_{-i}) = H_i\,(q_i,q_{-i})$$

so that G^t = G. Yet when t = 0 we have

$$H_i^0\,(q_i,q_{-i}) = H_i\,(q_i,p_{-i}).$$

Thus G^0 is a rather special game in which the payoff H_i^0 of each i will depend only upon his own strategy q_i and will be independent of the other players' strategies q_{-i}. Thus G^0 decomposes into n mutually separate maximization problems, one for each player.It can be shown that in most all cases of interest herein, the separable game G^0 has one equilibrium point (which in fact is a strong equilibrium point in pure strategies); and that moreover this equilibrium point q^o is identical to the best-reply combination q^o prescribed by naive Bayesian rationality (recall (2.1) above).

In discussing the games G^t and in comparing them with the original game G, it will be helpful to make use of the following terminology. Instead of saying that $q_i{*}$ is a best reply to q_{-i} in game G^t, we shall simply state that $q_i{*}$ is a G^t-best reply to q_{-i}. Likewise, we shall simply say that a point q that is an equilibrium point of a game G^t is a G^t-equilibrium point.

By John Nash's fundamental theorem of 1951, it is known that every well-defined non-cooperative game G has at least one equilibrium point. Thus for any game G^t the set E^t of all equilibrium points will be nonempty. Let X = X(G,p) be the graph of the correspondence $t \to E^t$ where t lies between 0 and 1. X can be shown generally to consist of a collection of pieces of one-dimensional (algebraic) curves. Consider a point x of X. It will have the form x = (t,q) where q is an equilibrium point of G^t. HS call t the t-coordinate of x, and they call q the strategy part of x. Clearly, since $t \epsilon I = (0,1)$ (the closed unit interval) whereas $q \epsilon Q$ is a copy of the strategy space of G (or equivalently of G^t), we see that the graph X will always be a subset of the closed cylindrical set Y = I X Q.

Suppose that the graph X contains a path L connecting a point $x^0 = (0,q^o)$ corresponding to an equilibrium point q^o of the separable game G^0, with a point $x^1 = (1, q*)$ corresponding to an equilibrium point of the original game $G^1 = G$. Then HS call L a feasible path while x^0 and x^1 will be called the starting point and the end point, respectively, of the path L. Moreover, the strategy part q* of this end point x^1 is called the solution of the original game G as given by the path L.

A formal definition of the TP is now possible. It simply selects and defines as the solution to G the point q* obtained by "tracing" (i.e. by following) a feasible path L from its starting point x^0 to its endpoint x^1. The fundamental theorem of HS in this regard is that under suitable conditions for the linear TP to apply, the TP will follow one path L*, called the distinguished path of G. Since this path L depends on both G and p we shall write L = L(G,p). Moreover, HS have proved that for any given pair (G,p) for which the (linear) TP is well-defined, the procedure will always select a unique equilibrium point $q^* = (q_1^*,\dots,q_n^*) = T(G,p)$. In short, the TP is usually well-defined and it works.

This completes our abstract description of the TP. Before turning to an application in Chapter 3, we shall conclude the present Chapter with a decision theoretical and game theoretical interpretation of the HS procedure. Hopefully, this will assist the reader in appraising the conceptual soundness and the applicability of the TP.

2.3. A Decision/Game Theoretical Interpretation of the TP

In this concluding Section, we provide a brief decision theoretical and game theoretical interpretation of the TP. In 2.3.A we raise a couple of points comparing the TP with the Bayes operator in decision theory. Then in 2.3.B we present HS's own interpretation.

2.3.A. Comparison with the Bayes Operator

Earlier in Section 2 we have mentioned the sense in which the TP is analogous to the "updating" that goes on in statistics and decision theory more generally via Bayes rule. A couple of points might help flesh out this analogy. First, it should be noted that Bayes Theorem is not a "model" that was conceived to model real-world cognitive processes. Bayes Theorem simply follows from the elementary axioms and rules of probability theory. Indeed, until the primitives that are operated upon by Bayes Theorem were given a certain epistemological interpretation by modern statisticians and decision theorists, the formula had no particular meaning. Once substance is ascribed to the primitives, however, the formula then assumes a very real meaning: it shows how "prior" information can be updated to include "new information". One can go even further. If one accepts the Savage axioms of decision theory, then it can be asserted that Bayes rule specifies how one should update prior information. The TP is quite different. It does not follow from anything analogous to the axioms of probability. It is intended to be, and admitted to be a model of the cognitive convergence of players' conflicting expectations and intentions. While we feel the TP is a highly original and significant approach to the most fundamental problem in all of game theory, namely that of selecting a "best" equilibrium point in a game, we also feel HS should discuss where the TP might stand in a whole family of alternative approaches to this same problem. Are there possible axioms sets that would identify the TP as the only/best manner in which to bring about convergence of strategic intentions and expectations? As of yet we do not know. Related to this matter is the problem of the logical status of the TP. HS point out it is simply a model. But is it ultimately a normative model, a conditionally normative model, a descriptive model, a positive model, or some combination of these?

A second salient difference between the Bayes operator and the TP is to be found in the matter of the "finality" of the results that each produces. In a statistical decision problem, there is in principle no end to the amount of new information that is obtained (or could be obtained) by the decision maker. Hence there is no "terminus" to the operation of Bayes rule. It is applied forever, at least in principle. In stating this, we are talking about the general class of (statistical) decision problems, and not simply the "optimal stopping problem" which is indeed somewhat different. In contrast to this, there is a well-defined endpoint to the operation of the TP. When the end of the distinguished path L* is reached, and a strategy n-tuple q* is identified, the process is finished. The discrepancy between the prior information and expectations and the actual expectations and information at t = 1 has disappeared, as explained in 2.2. This distinction may be of interest if and when researchers investigate entire classes of updating processes.

2.3.B. HS's Decision/Game Theoretical Interpretation

HS view the TP as a model of a solution process. It is seen as a process of analysis, computation, expectation formation, and strategy choice. The process is one in which n players will come to adopt -- and will come to expect each other to adopt -- one particular strategy n-tuple q* as the solution to their decision problem. As time passes, the process models gradual changes that take place in i's own tentative strategy plans and in his expectations about the other players' likely strategies. At any given moment, his plans and his expectations will be determined by a specific point $x^t = (t,q)$ of the path L*. HS call this point x^t i's position point at time t. As the process operates, x^t will continuously move along L* starting at x^0 and ending at x^1.

At any given moment, where i's position point is x^t, his tentative strategy plan will be to use the strategy q_i prescribed by the strategy part $q = (q_1,..,q_i,..,q_n)$ of this point x^t. But he will know that as long as t is less than unity, the strategy part q of x^t may not in fact correctly indicate the final solution of the game.Indeed, with t less than unity, q will in general not even be an equilibrium point of the given game G. Consequently, i will know that, even if q_j is the strategy preferred by player j at the moment, there is no assurance either on his part or anyone else's part that q_j will in fact be j's final and actual strategy.

More formally, HS argue that i will assign only probability t to the hypothesis that the other (n-1) players' behavior will in fact correspond to the strategy combination q_{-i} predicted by the strategy part q of the position point x^t. He will reserve the remaining probability (1-t) to his original hypothesis -- the naive Bayesian hypothesis -- that the players' behavior will correspond to the original (naive) behavior p_{-i}. As HS further show, this implies that his actual expectations about the other players' behavior will correspond to the probability distribution $r_i(t,q) = t(q_{-i}) + t(q_{-i}) + (1-t)(p_{-i})$.

This in turn can be used to justify the assumption that i's tentative strategy plan at that moment will be to use strategy q_i. This is so because it can be shown that q_i will be i's best reply to the probability distribution r_i which, as just seen, represents his expectations about the others' behavior.

Thus at each moment of the solution process, the t-value associated with the position point x^t will measure the degree of confidence that each player i has in the tentative prediction provided by this point x^t, namely that the other players will use strategy q_{-i} as prescribed by the strategy part q of x^t. As the process moves from $t = 0$ to $t = 1$ all players will gradually move from a state of complete predictive uncertainty to a state of complete confidence in the behavior that will in fact be adopted.

At the start of the process, the complete uncertainty of i about $j \neq i$'s behavior expresses itself in i's exclusive reliance on the prior p_{-i}. He uses this prior in forming his expectation in that he will entertain a probability distribution $r_i(0,q^o)$ assigning probability 1 to p_{-i} while assigning probability 0 to the strategy combination $q_{-i}o$ associated with his initial position point x^0. In contrast to this, at the end of the process, he will be in a state of full predictive certainty because he will feel able to predict which strategies the others will use. This is seen in the fact that his probability distribution $r_i(1,q*)$ assigns unit probability to the (n-1) tuple $q_{-i}*$ prescribed by the solution $q*$ to G. In terms of HS's distinction between first and second-order information, at any position point $x^t = (t,q)$ the first-order information to which each player i will react consists of the probability distribution $r_i(t,q) = t(q_{-i}) + (1-t)(p_{-i})$. In contrast to this, his second-order information consists of knowing that the others' tentative reactions to their own first-order information will be the strategy (n-1)-tuple q_{-i}. Formally, this second-order information can be defined as the probability distribution (q_{-i}) generated by q_{-i}.

3. A WORKED APPLICATION

In the previous chapter, an analytical overview of the HS TP was given. As promised, we now turn to a practical application of the TP. In Section 3.1 we describe the game to be solved. In 3.2 we discuss the question of how the prior probability vector p is determined, and we also discuss the use of the TP in defining certain "risk dominance" relationships that hold among different equilibrium points. In 3.3 we turn to numerical computations, and characterize the solution given by the TP. 3.4 is an interpretative conclusion.

3.1. A Two-Person Bargaining Game with Incomplete Information

Drawing on the recent results of HS, we shall analyze a one-parameter family of two-person bargaining games with incomplete information, namely G(w), where w is the parameter that is to be varied. Our ultimate purpose is to see what "solution" the TP yields for this family of games as the parameter w is varied. Let us now substantively describe this family G(w) of games.

In any given game G(w), there are two players I and II who must decide upon how to divide $100. Both players are assumed to have linear utility functions for money. For convenience, assume that these utility functions are normalized so that they ascribe x units of utility to $x. In the event that the players cannot reach agreement on how to divide the money, they receive the conflict payoffs c_I and c_{II}. We assume that $c_I = 0$ whereas $c_{II} = 0$ or w with w lying in the closed interval (\$0, \$50). Whether c_{II} is in fact equal to 0 or to w is determined by a chance move at the start of the game. More specifically these two possible conflict payoffs for player II each have probability 1/2. This probability distribution is assumed known to both players. However, the true value of c_{II} is actually only known to player II.

At this point, two observations are in order. First, the class of bargaining games G(w) we are analyzing is a subset of the class of games with incomplete information. This is so because, by definition, a game possesses incomplete information if at least one player does not know the utility payoffs of at least one antagonist. In the present case, player I does not know whether his opponent's (conflict) payoff is 0 or w. Second, it is now clear what the "parameter" is in the one-parameter family of games G(w) we are analyzing: namely, II's conflict payoff. For simplicity, we shall describe II as being of a weak type or of a strong type -- i.e. being in a weak or in a strong position -- according as whether $c_{II} = 0$ or w. By the end of the chapter we shall have seen how the solution prescribed by the TP changes as the parameter w changes*.

* HS themselves have investigated (and will publish) a more general class of games than that looked at herein. In their work, w can range from 0 all the way to 100. Guth and Selten (10) have analyzed the same class of problem from a mathematically different point of view.

A game G(w) is played in the following manner. Each player will choose a number x lying between 0 and 100, to be interpreted as the payoff proposed for player I. The number chosen by player I (by player II) will be denoted x_I (x_{II}). If $x_I = x_{II} = x$ then player I will receive the payoff $u_I = x$ whereas player II will receive the payoff $u_{II} = 100 - x$. On the other hand, in the event of conflict where $x_I \neq x_{II}$, the two players will only receive their conflict payoffs c_I and c_{II}.

At this point it is appropriate to transform the game G(w) described above into what HS call the agent normal form of the game. To understand this idea, recall that the "normal form" of any game is simply a description of the game in terms of (i) the players, (ii) the players' pure strategies, and (iii) the (utility) payoffs corresponding to any choice of a strategy n-tuple by the players. The "agent normal form" of a game is a straightforward generalization of the normal form. Specifically, each "player" is replaced by his "agents". It will be recalled from elementary game theory that a given player has as many agents as he has information sets. In our game, since player I has only one information set, he will have only one agent. But player II, who has two information sets, is replaced by two agents whom we shall call players 2 and 3. In the sequel, Roman numerals refer to the two-player representation of G(w) whereas Arabic numerals refer to the three-player representation.

It traditionally has been thought that a game could be fully described by its normal form. However, HS show in their work that it is necessary to use the more general agent normal form if certain critical information is to be retained in passing from the extensive form to a normal form. The game theoretical issues that arise concerning this point will not be discussed in the present paper.

In G(w), when player 1 selects a specific number $x_1 = x_I$ we shall say that he uses the pure strategy $s_1 = x_1$. Thus 1's pure strategy set is simply the closed interval $S_1 = (0,100)$. Likewise, a pure strategy s_2 of player 2 will be defined as a specific number $s_2 = x_2 = x_{II}$ that II may choose in the event that he is weak; and a pure strategy s_3 of player 3 will be defined as a specific number $s_3 = x_3 = x_{II}$ that player II may choose in the event he is strong. Thus 2's pure strategy set will likewise be the interval (0,100). However 3's strategy set S_3 will be the smaller interval (0,100-w). This is the case because if player 3 used any pure strategy $s_3 = x > 100-w$ then he would irrationally be accepting a payoff $u_3 = 100 - x < w$ which would be <u>less</u> than the payoff $u_3 = c_3 = w$ he could obtain by provoking conflict. In what follows, the payoff function $u_i = U_i(s_1, s_2, s_3) = U(s)$ will be written as such, where $s = (s_1, s_2, s_3)$. Also, we shall write $u = (u_1, u_2, u_3)$ and $u = U(s)$.

HS show that in the class of games G(w) under consideration, the equilibrium points will consist of all strategy combinations of the form $s = (s_1,s_2,s_3) = (x,x,x)$ with $0 \leqslant x \leqslant 100-w$. Any such equilibrium point (EP) amounts to an agreement between all three players, and will give a payoff vector $U(s) = (u_1,u_2,u_3) = (x\ 100-x,100-x)$*. What will be shown in the sequel is how the TP selects a unique EP from this infinity of EPs, and moreover, how this unique EP (the "solution") changes as the parameter w varies. But before proceeding with numerical computations, we shall discuss the problem of generating the prior probabilities p, and we shall introduce the concept of "risk dominance" as it arises in the HS theory.

3.2. Prior Probabilities and Risk Dominance in the HS Theory

In Chapter 2 we discussed the role of the prior probability distribution vector p in the HS theory. Let us now discuss how these prior probabilities are determined. Recall that a player's (e.g. player i's) prior probabilities express the likelihood he assigns to the adoption of different strategies at the beginning of the TP. In other words, the strategies implied by p constitute what we have

* There is in fact one other EP in a game G(w), and this is the strategy combination s = (100,0,0) reflecting complete disagreement between player I and II. However it can be shown that according to the TP, the "full agreement" EPs dominate this full disagreement outcome. We shall not discuss it further.

called "naive Bayesian behavior". More formally, for any player i $(i = 1,2,3)$ in $G(w)$, let $p_i = (p_i^1, p_i^2)$ be the probabilities that are assigned to the strategy doublet* s_i' and s_i''. Clearly $p_i^1 + p_i^2 = 1$.

In the HS theory, p_i is computed as follows. The theory makes the prior probability that player i will use strategy s_i' or strategy s_i'' proportional to his net payoffs associated with the corresponding equilibrium points s' and s'', respectively. Let us see how this happens at a more analytical level in the new theory.

Player i is chosen at random from a population B_i of potential players i, in which each potential player is characterized by a personal parameter b_i that is assumed uniformly distributed over the closed unit interval (0,1). Any player i, characterized by a specific value of b_i, will assume that the other two players will either both use their s'-strategies s_j' and s_k', or will both use their s"-strategies s_j'' and s_k'' -- with the former event having probability b_i whereas the latter event has the probability $(1-b_i)$. Denote

$$U_i(s_i',s_j',s_k') = u_i' \text{ and } U_i(s_i'',s_j'',s_k'') = u_i'' \qquad (3.1)$$

Also, by our assumptions about G(w) from Section 3.1, we have

* The reason why we are restricting ourselves here to a pair of strategies (rather than to an n-tuple) will be made clear in the sequel.

$$U_i(s_i'', s_j', s_k') = U_i(s_i', s_j'', s_k'') = c_i \tag{3.2}$$

Thus, player i's <u>expected</u> payoff, if he uses strategy s_i', will be

$$v_i' = b_i u_i' + (1 - b_i) c_i \tag{3.3}$$

while if he uses strategy s_i'' then his expected payoff will be

$$v_i'' = b_i c_i + (1 - b_i)\, u_i'' \tag{3.4}$$

By the expected utility principle of decision theory, player i will use strategy s_i' if $v_i' > v_i''$, that is (by simple algebraic substitution and rearrangement of terms) if

$$b_i \geqslant \frac{u_i'' - c_i}{(u_i' - c_i) + (u_i'' - c_i)} \tag{3.5}$$

And if this inequality is reversed, he will use strategy s_i''. Since the HS theory assumes that the population B_i is uniformly distributed over the unit interval as stated above, the probability that the former alternative will hold is clearly

$$p_i^1 = \frac{u_i' - c_i}{(u_i' - c_i) + (u_i'' - c_i)} \tag{3.6}$$

whereas the probability that the latter alternative will hold is

$$p_i^2 = \frac{u_i'' - c_i}{(u_i' - c_i) + (u_i'' - c_i)} \tag{3.7}$$

Here we see exactly how the HS theory makes the prior probabilities proportional to the net payoffs associated with the strategies.

One point that may trouble the reader at this junction is why HS make the assumption about the uniformity of the probability distribution over B_i, the set of possible "types" of player i. This is a matter that HS will explain in some detail in their forthcoming book. For the moment, it will hopefully suffice to have shown that use of this assumption (in the above derivations) yields the desired result: namely prior probabilities that are proportional to the net payoffs associated with the different strategy alternatives.

As we know from Chapter 2, the purpose of the TP itself is to determine a unique equilibrium point of the game as the "solution". In the present context, what this amounts to is determining whether strategy s' or s" is to be the solution. At this point we introduce the notion of risk dominance. In comparing and contrasting s' and s" via the theory of the TP, HS speak of the risk dominance relationships between s' and s" that are revealed by the workings of the TP. As we shall hopefully show, explicating the TP in terms of risk dominance adds intuitive insight into what is really going on. Let us now develop these points analytically in the remainder of 3.2.

Starting with the triplet $p = (p_1,p_2,p_3)$ of prior probability distributions, the TP itself is applied to an auxiliary family of games $G(w)^t$ for $0 \leqslant t \leqslant 1$. (Henceforth, we shall simply write G^t for $G(w)^t$.) The payoff functions for the auxiliary games are given by

$$U_i^t(s_i,s_j,s_k) = (1-t)\ U_i(s_i,p_j,p_k) + t\ U_i(s_i,s_j,s_k) \qquad (3.8)$$

as we know from equation (2.2). Let the set of equilibrium points of any game G^t be denoted E^t. Let P be the graph of the mapping $t \rightarrow E^t$. In each game G^t , we know from Chapter 2 that the players will use a strategy n-tuple s^t corresponding to a distinguished path L of the graph P. In the initial game G^0 corresponding to t = 0, this distinguished equilibrium point strategy n-tuple $s^0 = (s_1{}^0, s_2{}^0, s_3{}^0)$ is computed as follows. The initial strategy $s_i = s_i{}^0$ of player i is that strategy that maximizes his inital payoff, i.e.

$$U_i(s_i{}^0, p_j, p_k) = \text{MAX}\ (u_i{}'^0, u_i{}''^0) \qquad (3.9)$$

where

$$u_i{}'^0 = U_i(s_i{}', p_j, p_k) \quad \text{and} \quad u_i{}''^0 = U_i\ (s_i{}'', p_j, p_k) \qquad (3.10)$$

The maximization of U_i can be restricted to the two strategies $s_i{}'$ and s_i'' since all other strategies s_i will drive U_i to zero.

Let us now see what happens analytically as time goes on, i.e. as t becomes greater than zero and the tracing procedure unfolds. We shall be particularly interested in the conditions under which each of the players <u>deviates</u> from his initial (naive Bayesian) strategy choice: for it is in the analysis of these deviations that the notion of risk dominance is made clear.

At t = 0, suppose that the players' initial strategies are $s_i{}^0, s_j{}^0, s_k{}^0$. And more specifically, suppose that $s_i{}^0 = s_i{}'$. If the other two players now go on using the same strategies $s_j{}^0$ and $s_k{}^0$ at higher t values, then player i will not deviate from his initial strategy $s_i^0 = s_i'$ as long as the following inequality is maintained:

$$U_i^t(s_i', s_j^0, s_k^0) > U_i^t\ (s_i'', s_j^0, s_k^0). \qquad (3.11)$$

Likewise if the initial strategy $s_i^0 = s_i''$, then player i will stick to this as long as the converse inequality is preserved. In short, player i will only deviate from his initial strategy at that point $t = t_i$ when

$$U_i^t(s_i', s_j^0, s_k^0) = U_i^t(s_i'', s_j^0, s_k^0) \qquad (3.12)$$

or equivalently at that point $t = t_i$ when

$$\begin{aligned} &(1 - t_i)\ U_i(s_i', p_j, p_k) + t_i U_i(s_i', s_j^0, s_k^0) \\ = \ &(1 - t_i)\ U_i\ (s_i'', p_j, p_k) + t_i U_i\ (s_i'', s_j^0, s_k^0) \end{aligned} \qquad (3.13)$$

This important equation (3.13) yields a linear equation for computing the time (the t_i value) of deviation by i, if such a time exists at all. Applying like reasoning to players j and k, we can arrive at expressions for t_j and t_k. The conclusion we reach is that i will be the first to deviate if

$$t_i \leq \text{MIN}\ (t_j, t_k) \qquad (3.14)$$

provided only that t_i exists. We now apply the above machinery to the specific one-parameter family of games G(w) and see how the TP works in practice. The exercise will provide even greater insight into the notions of risk dominance and deviation-from-strategy.

3.3 Numerical Derivation of the Solution for the Family of Games G(w)

We now couple the foregoing analytical results with the numerical parameters of the family of games G(w) and show what the TP implies as the solution (distribution of $100 among the players). Let $s' = x'$ and $s'' = x''$ be two equilibrium points of G(w). Assume for convenience that $x' > x''$. Define z and v as

$$z = (x'+x'')/2 \qquad \text{and} \qquad v = x' - z \tag{3.15}$$

Transposing we can write

$$x' = z + v, \quad x'' = z - v, \quad \text{with} \quad v > 0 \tag{3.16}$$

Given these algebraic expressions, given the parameters of the family of games G(w), and given the analytical results of section 3.2, we know that the prior probabilities associated with s' and s'' are

$$p_1^{\;1} = x'/(x'+x'') = (z+v)/2z \qquad p_1^{\;2} = (z-v)/2z \tag{3.17}$$

$$p_2^{\;1} = (100-z-v)/200-2z) \qquad p_2^{\;2} = (100-z+v)/(200-2z) \tag{3.18}$$

$$p_3^{\;1} = (100-w-z-v)/(200-2w-2z) \qquad p_3^{\;2} = (100-w-z+v)/(200-2w-2z) \tag{3.19}$$

Let us now determine the conditions under which the three players will chose either strategy s_i' or s_i'' at the start of the game. Thereafter we analyze the deviation behavior of the players under various sets of circumstances.

From (3.17) and from the analysis of Section 3.2 above, we see that

$$u_1{}'^0 = U_1(s_1', p_2, p_3) = (z+v)\left(\frac{1}{2}\,\frac{100-z-v}{200-2z} + \frac{1}{2}\,\frac{100-w-z-v}{200-2w-2z} \right) \qquad (3.20)$$

whereas

$$u_1{}''^0 = U_1(s_1'', p_2, p_3) = (z-v)\left(\frac{1}{2}\,\frac{100-z+v}{200-2z} + \frac{1}{2}\,\frac{100-w-z+v}{200-2w-2z} \right) \qquad (3.21)$$

Player 1 will select strategy $s_1^0 = s_1'$ as his initial strategy if $u_1{}'^0 > u_1{}''^0$. But this is equivalent to the requirement that

$$4z^2 - (600-3w)z + (20{,}000 - 200w) > 0 \qquad (3.22)$$

This requirement will in turn be satisfied if

$$z < f \qquad (3.23)$$

where

$$f = (600-3w-\sqrt{40{,}000-400w+9w^2}\,)\ /\ 8 \qquad (3.24)$$

By like reasoning, player 1 will select $s_1^0 = s_1''$ when

$$z > f \qquad (3.25)$$

In the case of player 2, the above reasoning quickly yields that player 2 will choose $s_2^0 = s_2'$ as his initial strategy when $u_2{}'^0 > u_2{}''^0$ which in turn will obtain when

$$z < 50.$$

He will choose $s_2^0 = s_2''$ if $z > 50$. Finally, player 3 will choose $s_3^0 = s_3'$ when

$$z < 50 - (w/2) \qquad (3.26)$$

whereas he will select $s_3^0 = s_3''$ when (3.26) is reversed. Now, consider two important points about the quantity f originally defined in (3.24). First, it is seen that

$$50 - (w/2) < f < 50 \qquad (3.27)$$

Second, solving (3.24) gives the following values of f as a function of the conflict payoff parameter w (at $10 intervals):

$$\begin{array}{lllllll} w: & 0 & 10 & 20 & 30 & 40 & 50 \\ f: & 50 & 47 & 43.9 & 40 & 35.5 & 30.1 \end{array} \qquad (3.28)$$

Drawing upon the earlier results of Section 3.2 and upon equations (3.22)-(3,28) it will now be seen that the equilibrium point $s^* = f(w)$ is the solution to our one-parameter family of games. Or, to use the language of the TP, it will be seen that this particular equilibrium point risk dominates all other possible solutions.

The above results yield four possible sets of possible interrelationships between the parameters w, f, and z. Let us now see what the TP implies in each of these possible cases. Specifically, let us analyze which players deviate from their initial strategies (determined just above) and what this implies in each of the four cases.

Case 1: $z < 50 -(w/2)$: in this situation, $s^0 = (s_1^0, s_2^0, s_3^0) = (s_1', s_2', s_3') = s'$. s' is, we recall, a (strong) equilibrium point in the given game G(w). But when the players' initial strategies constitute a strong equilibrium point of the given game, then according to a HS lemma, they will go on using that strategy n-tuple during the duration of the TP. Hence in Case 1, s' risk dominates s''.

Case 2: $50 - (w/2) < z < f$. In this event, $s^0 = (s_1^0, s_2^0, s_3^0) = (s_1', s_2', s_3'')$. This strategy combination is clearly not an equilibrium point in the given game G(w); thus at least one player will deviate from his original strategy during the operation of the TP. Applying the theory and the formulas developed in Section 3.2, we find out that it is player 3 who will first deviate from his initial strategy $s_3^0 = s_3''$, thereupon adopting s_3'. The strategy triplet in effect at that switching time is -- in the same manner as we saw just above -- an equilibrium point of the given game. Hence it is followed until $t = 1$. So once again s' risk dominates s''.

Case 3: $f < z < 50$. Here $s^0 = (s_1'', s_2', s_3'')$. This is clearly not an equilibrium point of the given game. Via some rather complex calculations, HS show that player 3 has no incentive to deviate from his initial strategy. What is at issue is whether player 1 or player 2 will deviate first, i.e. the values of t_1 and t_2 as defined in Section 3.2. It turns out that who deviates first depends upon the numerical values of x' and x'' and also upon the value of w. However, what holds generally is that if $t_1 > t_2$ then s' risk dominates s'' whereas the converse holds if $t_1 < t_2$.

Case 4: $z > 50$. In this case, $s^0 = (s_1'', s_2'', s_3'') = s''$. This is of course an equilibrium point of the given game, so we see that s" risk dominates s'. The set of inequalities we have observed yields the desired result (among others), namely that the equilibrium point s* = (f,f,f) risk dominates all other equilibrium points.

3.4 An Intuitive Interpretation of the Solution to G(w)

At this point let us look at the numerical solution to G(w) given above by the TP, and let us examine how "reasonable" it is. In investigating this matter, we shall refer back to the classical Nash solution to the two-person bargaining problem referred to in Chapter 1. And in so doing, we shall compare the TP solution to G(w) with the Nash solution to a game that approximates G(w).

Let us first reproduce the numerical solution as originally given in (3.28):

w:	0	10	20	30	40	50	(3.28)
f:	50	47.2	43.9	40	35.5	30.1	

The solution s* = (f,f,f) tells us that all three players will agree that player 1 should receive f(w) whereas player two (or three, depending on II's true type) will receive \$100 - f(w). Thus, we see from (3.28) that if w = 0, each player gets \$50, if w = 10, player 1 gets \$47.20 whereas player II gets \$52.80, and so on. In sum, the higher the conflict payoff to player II, i.e.

the higher the value of w, then the lower the final payoff of money to player 1 whose conflict payoff is known to be 0. The TP yields an intuitively appealing result that corresponds to our sense about the ability of a "stronger" player to bargain down a "weaker" player -- where strength/weakness is given by the conflict payoff of player II.

Let us see how the TP solution relates to the classical Nash solution to the extent that such a comparison is possible. (Recall that Nash did not define a solution for two player games with incomplete information, so that strictly speaking he did not develop a theory that could solve G(w).) Understanding the relation between the TP solution and the classical Nash solution is particularly insightful since it was in analyzing the Nash solution that John Harsanyi (following F. Zeuthen) first developed his original theory of risk dominance.

In G(w) player 1 has an equal chance of facing player 2 or player 3 as his antagonist. Thus, as HS point out, our game G(w) lies half way between a two-person game G' in which player 1 would with certainty face player 2 as an opponent, and a game G" in which he would with certainty face player 3. By applying the classical Nash solution (or equivalently the new TP) to the true two-person game G' we find that each player receives $50. The intuitive reason for this of course is that G' would be a fully symmetrical game: both players have \$0 as their conflict payoffs, and their utility functions are identical since each player is assumed to have a function linear in money. Thus the Nash solution calls for the players to split the gains equally. Formally, $u_1' = u_2' = 50$. In G" players 1 and 3 receive $u_1'' = 50 - (w/2)$ and $u_3'' = 50 + (w/2)$. If we then think of the original game G as being a kind of "half-way house" between G' and G", it would seem that the players might receive as their payoffs

$$u_1* = (u_1' + u_1'')/2 = 50 - (w/4)$$

$$u_2* = u_3* = 100 - u_1* = 50 + (w/4).$$

Of course, the original game G is not in any formal sense a half-way house between G' and G". Rather it is a true two-player game with incomplete information. And as stated just above, the classical Nash theory did not deal with these games. Nonetheless it is interesting that in the one-parameter family G(w) of games we are considering, the TP solution is very close to the hypothetical payoffs u_1*, u_2*, and u_3* exhibited above. Specifically, we can see that

$$u_1(s*) = f < u_1* \quad \text{and} \quad u_2(s*) = u_3(s*) = 100 - f > u_2* = u_3*.$$

By simply substituting in various values of f(w) given in (3.28) the reader will immediately see that the TP results are in fact very close to the generalized Nashian results.

We feel that this result is important because of the well-known intuitive appeal of the Nash bargaining solution. Furthermore, it should be pointed out that if we apply the tracing procedure to any two-person bargaining game of complete information, we obtain as the solution to the game the classical Nash solution itself. What is particularly interesting to us about this fact is that while the classical theory and the new theory make use of quite different cognitive and psychological models (recall Section 1.2 of Chapter 1 above), both give identical results in the class of games for which the classical theory is well-defined.

3.5: Concluding Comments

In this Chapter we have analyzed a particular class of two-person games, namely two-person fixed threat bargaining games of incomplete information. Choice of this particular example has proven useful for several reasons. First, it has been possible to show how the tracing procedure works with a minimum of technical detail. In particular, it was not necessary to enter into a discussion of the concepts of the cells and formations of a game. Had we chosen a more complex example, this would have been necessary, and would have greatly lengthened and complicated the present paper. Second, the example permitted us to see how Harsanyi-Selten can take a game of incomplete information and represent it as a well-defined game of complete information by using their agent normal game form. Third, we have just seen how our example permitted a link to be established between the new tracing procedure theory and classical game theory. Thus we feel we have achieved our mission as stated in Chapter 1: to present a streamlined version of the tracing procedure theory that emphasizes its basic ideas, and to show how these ideas work in practice in the context of a concrete example. As pointed out in Section 1.4, a price paid for this particular mission is the omission of many technical considerations of great conceptual interest, in particular the notion of cells and formations, and the concept of a uniformly perturbed agent normal form. Nonetheless, the reader who has completed this research report should have gained a good understanding of the basic ideas of the new theory, their applicability, and their relationship to the only earlier determinate solution theory, namely the bargaining equilibrium model discussed in Section 1.3 of Chapter 1. If he has gained this, we have succeeded in our efforts.

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BIBLIOGRAPHICAL NOTE

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